COURSE NOTES

Reaching for Infinity

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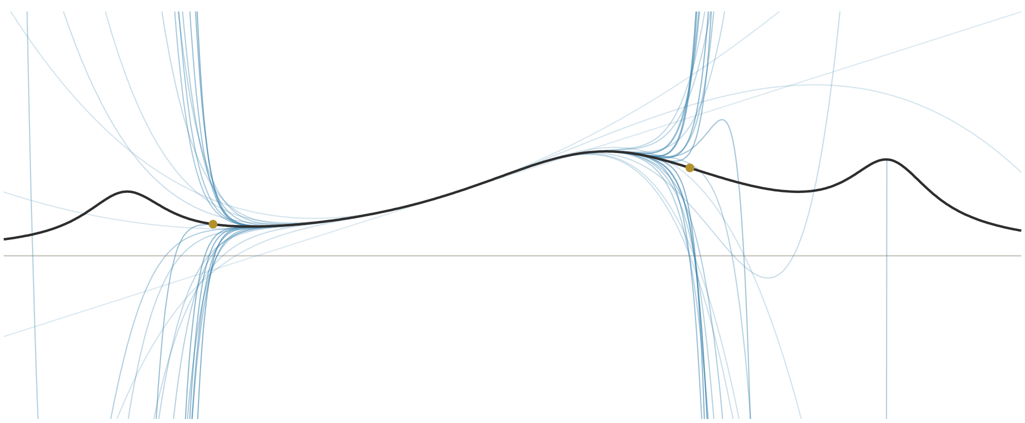

A first course in real analysis, told as the story of infinite processes: why we need them, how they fail, and the ideas — completeness, convergence, uniformity — that make them safe to use.

Contents

PART I

Numbers

CHAPTER 1

The Real Line

Stub chapter proving the build pipeline; real content gets rebuilt from the planning material (see `plan/MAP.md`).

1.1 Incompleteness of the Rationals

The diagonal of a unit square has length $\sqrt{2}$, and no fraction represents it exactly: the approximations

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \tag{1.1}$$

are each rational, each closer to $\sqrt{2}$ than the last, and their target is not in $\mathbb{Q}$.

Theorem 1.1 (Irrationality of the Square Root of Two). There is no rational number p/q with $(p/q)^2 = 2$.

Proof. Suppose p/q is in lowest terms and $p^2 = 2q^2$. Then p is even, so $p = 2k$ and $q^2 = 2k^2$, making q even too — contradicting lowest terms. ∎

1.2 Completeness

By theorem 1.1, the iteration (1.1) converges to no rational number: the real line is the completion that gives such processes a home.